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**ON THE EFFECT OF TRUNCATION IN SOME OR ALL COORDINATES  
OF A MULTINORMAL POPULATION**

by

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ON THE EFFECT OF TRUNCATION IN SOME OR ALL COORDINATES  
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Summary.

This paper is concerned with the following problem: Given a  $p$ -dimensional normal random variable with means zero, variances one, and correlation matrix  $R$ ; truncate this random variable in all coordinates, say at  $t_1, t_2, \dots, t_p$  respectively, and find expressions for  $E(X_1^m X_1^n)$  after truncation. An explicit solution of this problem is obtained for  $m = 1, 2$ ,  $n = 0$  and  $m = 1$ ,  $n = 1$ , that is for the expectations, variances and covariances of the distribution after truncation, and an extension of the method for greater values of  $m, n$  is indicated.

1. Introduction.

In various fields of applied statistics, such as psychological measurements and personnel selection, one frequently deals with populations which may be considered as originally

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multivariate normal, but modified by truncation in each coordinate separately. For example, a  $p$ -dimensional normal random variable  $X_1, X_2, \dots, X_p$  may represent  $p$  quantitative traits of an individual; very often an admission test requires that each of these traits be above a certain pre-assigned value, so that only those individuals pass the test for whom  $X_1 \geq t_1, \dots, X_p \geq t_p$ . It has been shown [1], that this method of selection has some undesirable properties; it is however frequently the only practical method, and hence it may be of some interest to study the properties of distributions obtained by such truncation.

In the present paper explicit expressions are obtained for the moments  $E(X_i)$ ,  $E(X_i^2)$ ,  $E(X_i X_j)$ , and it is indicated how the method can be extended to the general case of  $E(X_1^{\alpha_1} \dots X_p^{\alpha_p})$ . The possibility of truncation in some but not all coordinates is included since e.g. the case of  $X_1$  not truncated,  $X_2$  truncated at  $\tau$  corresponds to  $t_1 = -\infty$ ,  $t_2 = \tau$ . Explicit expressions are also obtained for the marginal probability density function of  $X_1$  and for the joint marginal p.d.f. of  $(X_1, X_2)$ , after truncation in  $X_1, X_2, \dots, X_p$ . Examples are given for the use of some of the results for determining  $t_1, t_2, \dots$  so that certain pre-assigned changes in the population are achieved.

## 2. A known lemma on determinants.

Let  $R$  be a  $p \times p$  matrix with the elements  $r_{ij}$ ; let  $R_{ij}$  be the cofactor of  $r_{ij}$ ,  $R^*$  the matrix of the  $R_{ij}$ .

$M_{ij}$  the cofactor of  $R_{ij}$  in  $R^*$ , and  $M_{i,j,u,v}^*$  the  $(p-2)$  dimensional minor in  $R^*$  obtained by deleting the  $i$ -th and  $j$ -th rows and  $u$ -th and  $v$ -th columns. Then we have

$$(2.1) \quad |R^*| = |R|^{p-1}$$

$$(2.2) \quad M_{ij} = r_{ij} |R|^{p-2}$$

$$(2.3) \quad M_{i,j,u,v}^* = (-1)^{u+v+i+j} (r_{iu} r_{jv} - r_{ju} r_{iv}) |R|^{p-3}.$$

The proof of this lemma may be found in standard treatises on determinants, e.g. [5] p.31.

### 3. Equations for the moments $E(X_1^m X_2^n)$ .

We consider a multi-normal  $p$ -dimensional random variable  $X_1, X_2, \dots, X_p$ , with the correlation matrix  $R = (r_{ij})$ ,  $i, j = 1, 2, \dots, p$ , and (without loss of generality) the means 0 and variances 1. Its p.d.f. is

$$(3.1) \quad f(X_1, X_2, \dots, X_p) = \frac{1}{(2\pi)^{p/2} \sqrt{|R|}} \exp \left\{ -\frac{1}{2} \sum_{i,j=1}^p \frac{R_{ij}}{|R|} X_i X_j \right\}.$$

The distribution is assumed to be non-singular and hence the

quadratic form  $\sum_{i,j=1}^p \frac{R_{ij}}{|R|} X_i X_j$  is positive definite.

Truncating  $X_1, X_2, \dots, X_p$  at  $t_1, t_2, \dots, t_p$  respectively we have for the new p.d.f. of  $X_1, X_2, \dots, X_p$  after truncation

$$(3.2) \quad g(x_1, x_2, \dots, x_p) = \begin{cases} f(x_1, x_2, \dots, x_p) & \text{for } x_j \geq t_j, j=1, 2, \dots, p \\ 0 & \text{elsewhere.} \end{cases}$$

The following notations will be used in the rest of the paper:

$$(3.21) \quad \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}; \quad \Phi(t) = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-s^2/2} ds$$

$$(3.22) \quad c_p(t_1, t_2, \dots, t_p; R) =$$

$$= \frac{1}{(2\pi)^{p/2} \sqrt{R!}} \int_{t_1}^{\infty} \dots \int_{t_p}^{\infty} e^{-\frac{1}{2} \sum_{i,j=1}^p \sum_{k=1}^R x_i x_j} dx_1 \dots dx_p$$

With these notations (3.2) becomes

$$(3.3) \quad g(x_1, x_2, \dots, x_p) =$$

$$= \begin{cases} c_p^{-1}(t_1, t_2, \dots, t_p; R) f(x_1, x_2, \dots, x_p) & \text{for } x_j \geq t_j, j=1, 2, \dots, p \\ 0 & \text{elsewhere.} \end{cases}$$

To obtain  $E(x_1^m x_2^n)$  for  $m \neq n$ , we set, again without loss of generality,  $m=1, n=2$  and write

$$(3.4) \quad E(x_1^m x_2^n) = \frac{G_p^{-1}(t_1, \dots, t_p; R)}{(2\pi)^{p/2} \sqrt{|R|}} \int_{t_1}^{\infty} \dots$$

$$\dots \int_{t_p}^{\infty} x_1^m x_2^n e^{-\frac{1}{2} \sum_{i,j=1}^p \frac{R_{ij}}{|R|} x_i x_j} dx_1 \dots dx_p =$$

$$= \frac{G_p^{-1}(t_1, \dots, t_p; R)}{(2\pi)^{p/2} \sqrt{|R|}} \int_{t_1}^{\infty} x_1 e^{-\frac{R_{11}}{2|R|} x_1^2} x_1^{m-1} \int_{t_2}^{\infty} \dots$$

$$\dots \int_{t_p}^{\infty} x_2^n e^{-\frac{1}{2|R|} (2x_1 \sum_{i=2}^p R_{i1} x_i + \sum_{i,j=2}^p R_{ij} x_i x_j)} dx_2 \dots dx_p dx_1 =$$

Integrating the right side by parts, we obtain from (3.4) after some simplification

$$(3.5) \quad R_{11} E(x_1^m x_2^n) + \sum_{i=2}^p R_{i1} E(x_1 x_1^{m-1} x_2^n) - |R| E \left[ \frac{d}{dx_1} (x_1^{m-1} x_2^n) \right] =$$

$$= \frac{G_p^{-1}(t_1, \dots, t_p; R)}{(2\pi)^{p/2} \sqrt{|R|}} |R| t_1^m e^{-\frac{R_{11} t_1^2}{2|R|}} \int_{t_2}^{\infty} \dots$$

$$\dots \int_{t_p}^{\infty} x_2^n e^{-\frac{1}{2|R|} (2t_1 \sum_{i=2}^p R_{i1} x_i + \sum_{i,j=2}^p R_{ij} x_i x_j)} dx_2 \dots dx_p$$

To evaluate the integral on the right side we apply the transformation

$$(3.6) \quad v_1 = \frac{x_1 - r_{11}t_1}{\sqrt{1 - r_{11}^2}}; \quad i = 2, 3, \dots, p$$

and obtain, using the lemma of Section 2,

$$(3.7) \quad R_{11}E(x_1^n x_2^n) + \sum_{i=2}^p R_{1i}E(x_1^{n-1} x_2^n) = R_{11}E\left(\frac{\partial}{\partial x_1}(x_1^{n-1} x_2^n)\right) =$$

$$= \frac{\varphi(t_1) R_{11} t_1^{n-1} \sqrt{1 - r_{11}^2}}{G_p(t_1, \dots, t_p; R) (2\pi)^{\frac{p-1}{2}}} \int_{\frac{t_2 - r_{21}t_1}{\sqrt{1 - r_{21}^2}}}^{\infty} \dots$$

$$\dots \int_{\frac{t_p - r_{p1}t_1}{\sqrt{1 - r_{p1}^2}}}^{\infty} (\sqrt{1 - r_{21}^2} v_2 + r_{21}t_1)^n e^{-\frac{1}{2} \sum_{i,j=2}^p \rho_{ij} v_i v_j} dv_2 \dots dv_p$$

where

$$(3.8) \quad \rho_{ij} = \frac{R_{ij}}{R_{11}} \sqrt{(1 - r_{11}^2)(1 - r_{j1}^2)}, \quad i, j = 2, 3, \dots, p.$$

The matrix  $(\rho_{ij})$  is positive definite since  $\left(\frac{R_{ij}}{R_{11}}\right)$  was assumed positive definite.

Replacing the subscripts 1,2 by w,s respectively we obtain

$$(3.9) \quad R_{ww} E(X_w^m X_s^n) + \sum_{\substack{i=1 \\ i \neq w}}^p R_{iw} E(X_i^{m-1} X_s^n) - |R| E \left[ \frac{d}{dX_w} (X_w^{m-1} X_s^n) \right] =$$

$$= \frac{\varphi(t_w) |R| t_w^{m-1} \sqrt{\rho(w)}}{g_p(t_1, \dots, t_p; R) (2\pi)^{\frac{p-1}{2}}} \int_{\frac{t_p - r_{pw} t_w}{\sqrt{1-r_{pw}^2}}}^{\infty} \dots$$

$$\dots \int_{\frac{t_p - r_{pw} t_w}{\sqrt{1-r_{pw}^2}}}^{\infty} (1-r_{sw}^2 v_s + r_{sw} t_w)^n e^{-\frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq w}}^p \rho_{ij}^{(w)} v_i v_j} \prod_{\substack{i=1 \\ i \neq w}}^p dv_i$$

for  $s, w=1, 2, \dots, p$ ,

where

$$(3.91) \quad \rho_{ij}^{(w)} = \frac{R_{ij}}{T_{ij}} \sqrt{(1-r_{iw}^2)(1-r_{jw}^2)}, \quad i, j=1, 2, \dots, p.$$

It can be verified that the inverse of the matrix  $(\rho_{ij}^{(w)})$  is

$$(3.92) \quad (\rho_{ij}^{(w)})^{-1} = \left( \frac{T_{ij} - T_{iw} T_{jw}}{\sqrt{(1-r_{iw}^2)(1-r_{jw}^2)}} \right) = T_{ij}, \quad \text{for } i, j=1, 2, \dots, p,$$

that is the matrix of partial correlation coefficients

$$r_{ij:w} = 1, j=1, 2, \dots, p.$$

4. Special cases:  $m=1, n=0$ ;  $m=2, n=0$ ;  $m=n=1$ .

Letting  $m=1, n=0$  in (3.9) we obtain

$$(4.1) \quad \sum_{i=1}^p R_{i:w} E(X_i) = \frac{\varphi(t_w) |R| \sqrt{\rho_{11}^{(w)}}}{\sigma_p(t_1, \dots, t_p; R) (2\pi)^{\frac{p-1}{2}}} \int_{-\infty}^{\infty} \frac{t_1 - r_{1w} t_w}{\sqrt{1 - r_{1w}^2}} \dots$$

$$\dots \int_{-\infty}^{\infty} \frac{t_p - r_{pw} t_w}{\sqrt{1 - r_{pw}^2}} \dots \prod_{j=1}^p dv_j$$

for  $w=1, 2, \dots, p$ .

Using the abbreviation

$$(4.2) \quad h(w) = \frac{\varphi(t_w) |R| \left( \frac{t_1 - r_{1w} t_w}{\sqrt{1 - r_{1w}^2}}, \dots, \frac{t_p - r_{pw} t_w}{\sqrt{1 - r_{pw}^2}}; t_w \right)}{\sigma_p(t_1, \dots, t_p; R)}$$

we can rewrite the equations (4.1) more concisely, as

$$(4.3) \quad \sum_{i=1}^p R_{i:w} E(X_i) = h(w) \cdot |R|, \quad w=1, 2, \dots, p.$$

To solve this system of equations we use (2.1) and (2.2) and obtain

$$(4.4) \quad E(X_w) = \sum_{i=1}^p r_{iw} h(i), \quad w=1,2,\dots,p.$$

Next, to obtain equations for the second moments, we set  $m=2$ ,  $n=0$ , and  $m=1$ ,  $n=1$  in (3.9). Using the notation

$$h(w) = \frac{\phi(t_w) \sqrt{\rho(w)}}{G_p(t_1, \dots, t_p; R) (2\pi)^{\frac{p-1}{2}}} \int_{-\infty}^{\infty} \dots$$

$$\dots \frac{\int_{t_w - r_{pw}}^{t_w + r_{pw}} \dots \sqrt{1-r_{pw}^2} \dots \phi(t_w) \sqrt{\rho(w)} \dots \prod_{i=1}^p dv_i$$

we obtain, respectively,

$$(4.5) \quad \sum_{i=1}^p R_{iw} E(X_i X_w) = [R] = R_w h(w), \quad w=1,2,\dots,p, \text{ and}$$

$$(4.7) \quad \sum_{i=1}^p R_{iw} E(X_i X_w) = R_w h(w), \quad w=1,2,\dots,p, \quad w \neq i$$

Combining (4.6) and (4.7), we have

$$(4.8) \quad \sum_{i=1}^p R_{is} N(X_i, X_s) = |R| [\sqrt{w_s} + h(ws)], \quad s=1,2,\dots,p$$

This is a system of  $p^2$  equations in the  $\frac{p(p+1)}{2}$  unknowns

$E(X_s X_s) = E(X_s X_s)$ . Since  $p^2 \geq \frac{p(p+1)}{2}$ , it will be sufficient

to choose a subsystem of  $\frac{p(p+1)}{2}$  independent equations. The

equations for which  $1 \leq s \leq w \leq p$  form such a system; to

show this, we arrange these equations and their unknowns in the

manner indicated by the following table:

	$E(X_1^2)$	$E(X_1 X_2)$	$E(X_1^2)$	$E(X_1 X_3)$	$E(X_2 X_2)$	$E(X_3^2) \dots E(X_1 X_p)$	$\dots \dots E(X_p^2)$	$ R  [h(11)+1]$
	$R_{11}$	$R_{12}$	0	$R_{13}$	0	0 ... $R_{1p}$	0 ... 0	$ R  h(12)$
	0	$R_{11}$	$R_{12}$	0	$R_{13}$	0 ... 0	$R_{1p}$ 0 ... 0	$ R  [h(22)+1]$
	0	$R_{21}$	$R_{22}$	0	$R_{23}$	0 ...	$R_{2p}$ 0 ... 0	
(4.9)	.....							
	.....							
	.....							
	.....							$ R  h(1p)$
	0					$R_{1p}$	$R_{1p}$	
	0							
	.....							$ R  [h(pp)+1]$
	0					$R_{pp}$	$R_{pp}$	

In (4.9) all the columns except the last contain the coefficients of the unknown indicated in the column heading, while the last column (headed  $K$ ) contains the right side terms of those equations of system (4.6) for which  $1 \leq s \leq w \leq p$ . The determinant of the coefficient matrix in (4.9) is

$$(4.10) \quad \Delta = R_{11} \cdot \begin{vmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{vmatrix} \cdots \begin{vmatrix} R_{11} & \cdots & R_{1p} \\ \vdots & & \vdots \\ R_{p1} & \cdots & R_{pp} \end{vmatrix}$$

and is  $\neq 0$  since each factor is a principal minor of a positive definite matrix. Thus these equations yield a solution for (4.8). Using (2.1) and (2.2) it is easily verified that

$$(4.11) \quad \bar{E}(\bar{x}, \bar{x}_s) = \sum_{i=1}^p r_{si} [h(ij) + \int_{ij}] , \quad j, s = 1, 2, \dots, p .$$

In evaluating  $h(ws)$  we can use the previous results, in particular (4.4), on the first term

$$\frac{\varphi(t_w) \sqrt{|p_{11}^{(w)}|}}{G_p(t_1, \dots, t_p; R) (2\pi)^{\frac{p-1}{2}}} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2} t_1^2 - \frac{1}{2} t_w^2}}{\sqrt{1 - r_{1w}^2}} \cdots$$

$$\dots \int_{\frac{t_p - r_{pw} t_w}{\sqrt{1-r_{pw}^2}}}^{\infty} \sqrt{1-r_{sw}^2} v_s e^{-\frac{1}{2} \sum_{i,j=1}^p \sum_{i \neq j} \rho_{ij}^{(w)} v_i v_j} \prod_{i=1}^p dv_i,$$

for it represents, except for appropriate constants, the marginal expectation of the coordinate  $v_s$  of a  $(p-1)$  dimensional random variable  $v_1, v_2, \dots, v_{w-1}, v_{w+1}, \dots, v_p$ ,

$$\text{truncated at } \frac{t_1 - r_{1w} t_w}{\sqrt{1-r_{1w}^2}}, \frac{t_2 - r_{2w} t_w}{\sqrt{1-r_{2w}^2}}, \dots, \frac{t_{w-1} - r_{(w-1)w} t_w}{\sqrt{1-r_{(w-1)w}^2}},$$

$$\frac{t_{w+1} - r_{(w+1)w} t_w}{\sqrt{1-r_{(w+1)w}^2}}, \dots, \frac{t_p - r_{pw} t_w}{\sqrt{1-r_{pw}^2}}, \text{ respectively; the second term}$$

$$\frac{\phi(t_w) \sqrt{\rho_{11}^{(w)}}}{G_p(t_1, \dots, t_p; R) (2\pi)^{\frac{p-1}{2}}} \int_{\frac{t_1 - r_{1w} t_w}{\sqrt{1-r_{1w}^2}}}^{\infty} \dots$$

$$\dots \int_{\frac{t_p - r_{pw} t_w}{\sqrt{1-r_{pw}^2}}}^{\infty} r_{sw} t_w e^{-\frac{1}{2} \sum_{i,j=1}^p \sum_{i \neq j} \rho_{ij}^{(w)} v_i v_j} \prod_{i=1}^p dv_i$$

$$= \frac{g(t_w) r_{pw} t_w}{G_p(t_1, \dots, t_p; R)} G_{p-1} \left( \frac{t_1 - r_{1w} t_w}{\sqrt{1 - r_{1w}^2}}, \dots, \frac{t_p - r_{pw} t_w}{\sqrt{1 - r_{pw}^2}}; T_w \right).$$

Expressions (4.4) and (4.11) appear to be useful for setting up and solving various practical problems of the kind illustrated in Section 6. The numerical evaluation of these expressions requires the computation of integrals of the type (3.22). The values of such integrals for  $p=2$  may be found in Pearson's Table VIII - IX in [6]. For  $p=3$  and  $p=4$  a large number of the required integrals may still be found in these tables. For  $p \geq 4$  all such integrals involved have to be calculated, a task which may require the use of high-speed computing equipment.

To obtain values of higher moments, one must go back to (3.9), and by similar manipulations as above, obtain the required number of independent equations to solve for the unknowns.

5. Expressions for the marginal distributions of  $X_1$  and  $(X_1, X_2)$  after truncation in  $X_1, X_2, \dots, X_p$ .

If  $\psi_1(X_1)$  is the p.d.f. of  $X_1$  after truncation in  $X_1, X_2, \dots, X_p$ , then by (3.3)

$$(5.1) \quad \psi_1(x_1) = g^{-1}(t_1, \dots, t_p; R) \int_{t_2}^{\infty} \dots \int_{t_p}^{\infty} f(x_1, \dots, x_p) dx_2 \dots dx_p.$$

Using the transformation

$$X_1 = X_1$$

(5.2)

$$V_1 = \frac{X_1 - r_{11}X_1}{\sqrt{1 - r_{11}^2}}, \quad i=2,3,\dots,p$$

one obtains

$$E(V_j) = 0, \quad j=2,3,\dots,p$$

$$E(V_j^2) = 1, \quad j=2,3,\dots,p$$

(5.3)

$$E(V_i V_j) = \frac{r_{11} - r_{11}r_{ij}}{\sqrt{(1-r_{11}^2)(1-r_{ij}^2)}}, \quad i,j=2,3,\dots,p$$

$$E(X_1 V_j) = 0 \quad j=2,3,\dots,p$$

By [2], p.313,  $X_1, V_2, \dots, V_p$  is again distributed according to the multi-normal law, and hence according to (5.2) and (5.3), expression (5.1) becomes:

$$(5.4) \quad \psi_1(X_1) = \frac{\phi(X_1)}{G_p(t_1, \dots, t_p; R)} G_{p-1}\left(\frac{t_2 - r_{21}X_1}{\sqrt{1-r_{21}^2}}, \dots, \frac{t_p - r_{p1}X_1}{\sqrt{1-r_{p1}^2}}; T_1\right)$$

where  $T_1$  is defined in (3.92).

If  $\psi_2(X_1, X_2)$  denotes the p.d.f. of  $X_1, X_2$  after truncation in  $X_1, X_2, \dots, X_p$ , then by (3.3)

$$(5.5) \quad \psi_2(X_1, X_2) = G^{-1}(t_1, \dots, t_p; R) \int_{t_3}^{\infty} \dots \int_{t_p}^{\infty} f(X_1, \dots, X_p) dX_3 \dots dX_p$$

Using the transformation

$$X_1 = X_1$$

$$X_2 = X_2$$

$$V_i = \frac{1}{\sqrt{\Delta_{11} \Delta_i}} [\Delta_{1i} X_1 + \Delta_{1i} X_2 + \Delta_{2i} X_2], \quad i=3,4,\dots,p,$$

where

$$\Delta_i = \begin{vmatrix} 1 & r_{12} & r_{1i} \\ r_{21} & 1 & r_{2i} \\ r_{i1} & r_{i2} & 1 \end{vmatrix}, \quad i=3,4,\dots,p$$

and  $\Delta_{st}$  is the cofactor of  $r_{st}$  in  $\Delta_i$ , one easily verifies that

$$E(V_i) = 0$$

$$E(V_i^2) = 1$$

$$(5.7) \quad E(V_i X_1) = 0 \quad i=3,4,\dots,p$$

$$E(V_i X_2) = 0$$

$$E(V_i V_j) = \frac{\Delta_{11}}{\sqrt{\Delta_{11} \Delta_i \Delta_{jj} \Delta_j}} [\Delta_{jj} r_{1i} + \Delta_{1j} r_{i1} + \Delta_{2j} r_{2i}],$$

$$i, j=3,4,\dots,p.$$

Hence, since  $(X_1, X_2, V_3, \dots, V_p)$  again has a multivariate normal distribution, we obtain from (5.6) and (5.7)

$$(5.8) \psi_2(x_1, x_2) = \frac{1}{2\pi\sqrt{1-r_{12}^2}} \cdot \frac{1}{2(1-r_{12}^2)} (x_1^2 - 2r_{12}x_1x_2 + x_2^2) \cdot \frac{g_{p-2}(y_3, \dots, y_p; S)}{g_p(t_1, \dots, t_p; R)},$$

where

$$y_i(t_i, x_1, x_2) = \frac{1}{\sqrt{\Delta_{11}\Delta_i}} [\Delta_{1i}t_i + \Delta_{1i}x_1 + \Delta_{2i}x_2], \quad i=3, 4, \dots, p$$

$$S = (s_{ij}) = (E(V_i V_j)).$$

#### 6. Some Applications.

The following problem is of practical interest: for a bivariate normal random variable  $(X_1, X_2)$  with expectations 0, variances 1 and known correlation coefficient  $r$ , it is required to find  $t_1$  and  $t_2$  so that, after truncation at  $t_1$  and  $t_2$ , the expectations of  $X_1$  and  $X_2$  assume the pre-assigned values  $m_1$  and  $m_2$ .

To find such  $t_1, t_2$ , we have according to (4.4)

$$(6.1) \quad m_1 = h(1) + r h(2)$$

$$(6.2) \quad m_2 = h(2) + r h(1).$$

Using expression (4.2) for  $h(i)$  and simplifying, one obtains:

$$(6.3) \quad L_1(t_1, t_2) = \frac{\phi(t_2) \phi\left(\frac{t_1 - r t_2}{\sqrt{1-r^2}}\right)}{\phi(t_1) \phi\left(\frac{t_2 - r t_1}{\sqrt{1-r^2}}\right)} = \frac{m_2 - r m_1}{m_1 - r m_2}$$

$$(6.4) \quad L_2(t_1, t_2) = \frac{\phi(t_1) \phi\left(\frac{t_2 - rt_1}{\sqrt{1-r^2}}\right)}{G_2(t_1, t_2; r)} = \frac{m_1 - rm_2}{1 - r^2}.$$

These equations show that the inequalities  $rm_1 < m_2 < \frac{1}{r} m_1$  are necessary for the existence of a solution.

To obtain numerical values for  $t_1$ ,  $t_2$ , one may consider (6.3) <sup>and (6.4)</sup> as equations of two curves in the  $t_1, t_2$  plane, and determine their intersection. The following numerical example will serve to illustrate the procedure:

Given  $r = .60$  and the required expectations after truncation  $m_1 = 1.5$ ,  $m_2 = 2.0$ . The right-hand sides in (6.3) and (6.4) become, respectively, 3.67 and 0.469. By trial, using tables, one finds the following three points on each of the curves:

$t_1$	$t_2$	$L_1(t_1 t_2)$	$t_1$	$t_2$	$L_2(t_1 t_2)$
.55	1.532	3.67	.50	1.430	.469
.60	1.594	3.67	.55	1.530	.469
.65	1.655	3.67	.60	1.626	.469

Plotting these values one finds for the point of intersection  $t_1 = .554$ ,  $t_2 = 1.537$ , and substituting these values into (6.1) and (6.2) one obtains  $m_1^* = 1.501$ ,  $m_2^* = 2.001$  which is a good approximation to the required values.

Next we consider the following problem: We wish to truncate  $X_1$  and  $X_2$  at  $t_1$  and  $t_2$  respectively so that the expectation of  $X_1$  after truncation has a pre-assigned value  $m_1$  and the retained part of the population

$$G(t_1, t_2; r) = \frac{1}{2\pi\sqrt{1-r^2}} \int_{t_1}^{\infty} \int_{t_2}^{\infty} \frac{1}{2(1-r^2)} (x^2 - 2rxy + y^2) dx dy$$

is as large as possible. This is equivalent to maximizing  $G(t_1, t_2; r)$  under the condition (6.1). Using Lagrange multipliers, we consider

$$H(t_1, t_2) = G(t_1, t_2; r) + \lambda [m_1 G(t_1, t_2; r) - \varphi(t_1) \phi\left(\frac{t_2 - rt_1}{\sqrt{1-r^2}}\right) - r \varphi(t_2) \phi\left(\frac{t_1 - rt_2}{\sqrt{1-r^2}}\right)] ;$$

we wish to solve the equations

$$\frac{\partial H}{\partial t_1} = 0, \quad \frac{\partial H}{\partial t_2} = 0, \quad \frac{\partial H}{\partial \lambda} = 0.$$

It is easily verified that these equations become, respectively,

$$(6.5) \quad \lambda = \frac{1}{t_1 - m_1}$$

$$(6.6) \quad \frac{\varphi\left(\frac{t_1 - rt_2}{\sqrt{1-r^2}}\right)}{\phi\left(\frac{t_1 - rt_2}{\sqrt{1-r^2}}\right)} = \frac{1 + m_1 \lambda - \lambda r t_2}{\lambda \sqrt{1-r^2}}$$

$$(6.7) \quad m_1 G(t_1, t_2; r) - \varphi(t_1) \phi\left(\frac{t_2 - rt_1}{\sqrt{1-r^2}}\right) - r \varphi(t_2) \phi\left(\frac{t_1 - rt_2}{\sqrt{1-r^2}}\right) = 0.$$

From (6.5) and (6.6) we obtain:

$$(6.8) \quad \frac{\phi\left(\frac{t_1 - rt_2}{\sqrt{1-r^2}}\right)}{\phi\left(\frac{t_1 - rt_2}{\sqrt{1-r^2}}\right)} = \frac{t_1 - rt_2}{\sqrt{1-r^2}}.$$

It is well known [3], [4], that for finite  $U > 0$ ,  $\frac{\phi(U)}{\phi(U)} > U$ . From (6.8) we see, therefore, that our problem has no solution with  $t_1$  and  $t_2$  both finite. The following four possibilities remain:

- a)  $r > 0$ ,  $t_1 = -\infty$ ,  $t_2 < \infty$
- b)  $r > 0$ ,  $t_1 < \infty$ ,  $t_2 = -\infty$
- c)  $r < 0$ ,  $t_1 = -\infty$ ,  $t_2 < \infty$
- d)  $r < 0$ ,  $t_1 < \infty$ ,  $t_2 = -\infty$ .

In the cases a) and c) (6.7) yields:  $\frac{\phi(t_2)}{\phi(t_2)} = \frac{m_1}{r}$ ,

while in cases b) and d) (6.7) becomes  $\frac{\phi(t_1)}{\phi(t_1)} = m_1$ . Since

$\frac{\phi(U)}{\phi(U)}$  is a monotonic function of  $U$ , increasing from 0 to

$\infty$ , we reach the following conclusions:

For  $r > 0$ , (cases a) and b) ),  $\max G(t_1, t_2; r)$  under condition  $E(X_1) = m_1$  is obtained by truncating in  $X_1$  alone

at  $t_1$ , where  $t_1$  is obtained from  $\frac{\varphi(t_1)}{\phi(t_1)} = m_1$ . Only positive values of  $m_1$  can be achieved.

For  $r < 0$ , (cases c) and d), we must truncate in  $X_2$  alone and use the solution obtained from  $\frac{\varphi(t_2)}{\phi(t_2)} = \frac{m_1}{r}$ , for  $m_1 < 0$ ; and truncate in  $X_1$  alone, at  $t_1$  obtained from  $\frac{\varphi(t_1)}{\phi(t_1)} = m_1$ , for  $m_1 > 0$ . Tables of  $\frac{\varphi(U)}{\phi(U)}$  may be found e.g. in [6].

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